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STRESS–STRAIN STATE OF AN ANISOTROPIC PLATE WITH CURVILINEAR CRACKS AND THIN RIGID INCLUSIONS

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A mixed problem of linear elasticity for an infinite anisotropic plate with cuts and thin undeformable inclusions located along arbitrary open smooth curves is solved with the use of complex potentials. Special representations of the solutions are constructed and a governing system of singular integral equations is obtained. A numerical algorithm for determining the stress-strain state of the plate, including the stress-intensity factors at the tips of cuts and rigid inclusions, is proposed. Calculation results are given.

Introduction. In the structures in service, brittle fracture generally begins near the technological or constructional stress concentrators, in particular cuts or rigid pointed inclusions. One method of determining the elastic and limit equilibrium of deformable solids with pointed inclusions and of studying the mutual effect of closely located inclusions and cuts is the method of singular integral equations. Berezhnitskii, Panasyuk, and Stashchyuk [1] proposed a system of these equations to study the stress–strain state of an isotropic plate with a finite number of curvilinear rigid inclusions and cuts under various force actions. For some cases, they gave exact or asymptotic formulas (obtained with the use of the small parameter for large distances between the defects) and developed numerical algorithms for calculating the stress-intensity factors at the tips of cracks and inclusions. Sil'verstov and Shumilov [2] studied the stress–strain state of a packet of isotropic plates rigidly connected along the curves.

Maksimenko et al. [3–6] developed methods of analyzing the stress–strain state of anisotropic plates with complex cuts and of plates with cuts, holes, and rectilinear elastic locks. In this paper, the general representations of solutions for the problem of interaction between cuts and thin undeformable inclusions are constructed under the following assumptions: the objects are located along smooth open curves and do not touch each other and the contact between the cut edges is excluded. The problem is reduced to a system of singular integral equations effectively solved by numerical methods. The advantage of the proposed approach is the unified form of integral equations for the contours of cuts and inclusions, which simplifies the development of an algorithm of numerical solution. Results of numerical analysis of a number of new problems for anisotropic and isotropic (the limiting transition in anisotropy parameters is employed) plates are given and compared with the existing solutions. The high accuracy of the algorithm is shown.

Formulation of the Problem. We consider rectilinear-anisotropic elastic plate of constant thickness that occupies the plane $z = x + iy$. The uniformly distributed stresses σ_x^∞, σ_y^∞, and τ_{xy}^∞ are applied at infinity. The plate contains through cuts (cracks) and thin undeformable inclusions located along the smooth curves $L_j = (a_j, b_j)$ for $j = 1, \ldots, k_1$ and $j = k_1 + 1, \ldots, k$, respectively (Fig. 1);

$$L = \bigcup_{j=1}^{k} L_j, \quad L^{(1)} = \bigcup_{j=1}^{k_1} L_j, \quad L^{(2)} = \bigcup_{j=k_1+1}^{k} L_j.$$

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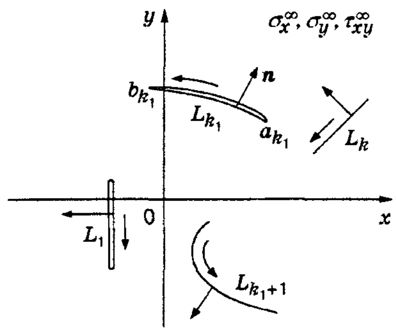

Fig. 1

For each curve, the normals $n(t)$ directed toward the right in moving from a_j to b_j are used. By assumption, the cut edges do not contact each other and are subjected to the action of self-balanced, uniformly distributed loads $P(t)$:

$$X_n^\pm(t) + iY_n^\pm(t) = \pm P(t), \qquad t \in L^{(1)}. \tag{1}$$

The curvilinear inclusions can move as a rigid body:

$$u^\pm(t) + iv^\pm(t) = g_1(t) + ig_2(t) = G(t), \quad t \in L^{(2)}; \quad G(t) = c_j + i\varepsilon_j t, \quad t \in L_j. \tag{2}$$

Here c_j is a complex constant and ε_j is the unknown or prescribed angle of rotation of the rigid inclusion L_j. The plus and minus signs refer to the left and right edges of the cut or inclusion, respectively.

The stress–strain state of the plane is to be determined.

Form of the Potentials. Let μ_1 and μ_2 be different roots of the characteristic equation [7] $a_{11}\mu^4 - 2a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0$, in which a_{ij} are the strain coefficients from Hooke's law. We assume that $\operatorname{Im}\mu_1 > 0$ and $\operatorname{Im}\mu_2 > 0$.

By analogy with [4], we seek the Lekhnitskii potentials in the form

$$\Phi_\nu(z_\nu) = \Phi_{\nu 0} + \Phi_{\nu 1}(z_\nu) + \Phi_{\nu 2}(z_\nu) \qquad (\nu = 1, 2). \tag{3}$$

Here $z_\nu = x + \mu_\nu y$ and $\Phi_{\nu 0}$ are known constants determined by the tractions at infinity for a defect-free plane [8],

$$\Phi_{\nu 1}(z_\nu) = \frac{1}{2\pi i}\int\limits_{L^{(1)}} \frac{\omega_\nu(\tau)\,d\tau_\nu}{\tau_\nu - z_\nu}, \qquad \Phi_{\nu 2}(z_\nu) = \frac{1}{2\pi i}\int\limits_{L^{(2)}} \frac{\mu_\nu(\tau)\,d\tau_\nu}{\tau_\nu - z_\nu},$$

$d\tau_\nu = (\mu_\nu \cos\varphi(\tau) - \sin\varphi(\tau))\,ds = M_\nu(\tau)\,ds$, $\varphi(\tau)$ is the angle between the normal $n(\tau)$ and the x axis, and ds is the differential of the arc length of the curve.

System of Integral Equations of the Problem. According to [3, 7], we write the boundary conditions (1) and (2) in the form

$$\begin{aligned} &a(t)\Phi_1^\pm(t_1) + b(t)\bar\Phi_1^\pm(t_1) + \Phi_2^\pm(t_2) = F^\pm(t), \qquad t \in L^{(1)}, \\ &A(t)\Phi_1^\pm(t_1) + B(t)\bar\Phi_1^\pm(t_1) + \Phi_2^\pm(t_2) = W^\pm(t), \qquad t \in L^{(2)}, \end{aligned} \tag{4}$$

where

$$a(t) = a_0\frac{M_1(t)}{M_2(t)}, \quad b(t) = b_0\frac{\bar M_1(t)}{M_2(t)}, \quad a_0 = \frac{\mu_1 - \bar\mu_2}{\mu_2 - \bar\mu_2}, \quad b_0 = \frac{\bar\mu_1 - \bar\mu_2}{\mu_2 - \bar\mu_2},$$

$$A(t) = A_0\frac{M_1(t)}{M_2(t)}, \quad B(t) = B_0\frac{\bar M_1(t)}{M_2(t)}, \quad A_0 = \frac{\bar p_2 q_1 - p_1\bar q_2}{\bar p_2 q_2 - p_2\bar q_2}, \quad B_0 = \frac{\bar p_2\bar q_1 - \bar p_1\bar q_2}{\bar p_2 q_2 - p_2\bar q_2},$$

$$F^\pm(t) = \pm\frac{X_n^\pm(t) + \bar\mu_2 Y_n^\pm(t)}{(\mu_2 - \bar\mu_2)M_2(t)}, \quad W^\pm(t) = W(t) = \left(\bar p_2\frac{dg_2}{ds} - \bar q_2\frac{dg_1}{ds}\right)\frac{1}{(\bar p_2 q_2 - p_2\bar q_2)M_2(t)},$$

$$p_\nu = a_{11}\mu_\nu^2 - a_{16}\mu_\nu + a_{12}, \quad q_\nu = a_{12}\mu_\nu + a_{22}\mu_\nu^{-1} - a_{26}, \quad \nu = 1, 2.$$

Since the loads are self-balanced, we have $F^+(t) = F^-(t) = F(t)$.

Using representations (3) and the Sochotsky–Plemelj formulas, from (4) we obtain the system of singular integral equations for determining the desired densities $\omega_1(t)$, $\omega_2(t)$, $\mu_1(t)$, and $\mu_2(t)$ and the relations for $\omega_1(t)$ and $\omega_2(t)$ on the cuts and $\mu_1(t)$ and $\mu_2(t)$ on the rigid inclusions:

$$\begin{aligned}
&\int_{L^{(1)}} \frac{\omega_1(\tau)}{\tau_1 - t_1}\, d\tau_1 + \int_{L^{(1)}} \omega_1(\tau) K_{11}(t,\tau)\, ds + \int_{L^{(1)}} \bar{\omega}_1(\tau) K_{12}(t,\tau)\, ds \\
&\quad + \int_{L^{(2)}} \mu_1(\tau) K_{13}(t,\tau)\, ds + \int_{L^{(2)}} \bar{\mu}_1(\tau) K_{14}(t,\tau)\, ds = f_1^{**}(t), \quad t \in L^{(1)}, \\
&\int_{L^{(2)}} \frac{\mu_1(\tau)}{\tau_1 - t_1}\, d\tau_1 + \int_{L^{(2)}} \mu_1(\tau) K_{21}(t,\tau)\, ds + \int_{L^{(2)}} \bar{\mu}_1(\tau) K_{22}(t,\tau)\, ds \\
&\quad + \int_{L^{(1)}} \omega_1(\tau) K_{23}(t,\tau)\, ds + \int_{L^{(1)}} \bar{\omega}_1(\tau) K_{24}(t,\tau)\, ds = f_2^{**}(t), \quad t \in L^{(2)};
\end{aligned} \tag{5}$$

$$\begin{aligned}
&a(t)\omega_1(t) + b(t)\bar{\omega}_1(t) + \omega_2(t) = 0, \qquad t \in L^{(1)}, \\
&A(t)\mu_1(t) + B(t)\bar{\mu}_1(t) + \mu_2(t) = 0, \qquad t \in L^{(2)},
\end{aligned} \tag{6}$$

where

$$f_1^{**}(t) = \frac{\pi i \bar{F}(t)}{\bar{b}(t)} - \pi i \Big[\frac{\bar{a}(t)}{\bar{b}(t)}\,\bar{\Phi}_{10} + \Phi_{10} + \frac{1}{\bar{b}(t)}\,\bar{\Phi}_{20}\Big], \qquad t \in L^{(1)};$$

$$f_2^{**}(t) = \frac{\pi i \bar{W}(t)}{\bar{B}(t)} - \pi i \Big[\frac{\bar{A}(t)}{\bar{B}(t)}\,\bar{\Phi}_{10} + \Phi_{10} + \frac{1}{\bar{B}(t)}\,\bar{\Phi}_{20}\Big], \qquad t \in L^{(2)}.$$

Here, the kernels $K_{ij}(t,\tau)$ are regular.

We supplement the system by the equations

$$\int_{L_j} \omega_1(\tau)\, d\tau_1 = 0 \quad (j = 1, \ldots, k_1), \qquad \int_{L_j} \mu_1(\tau)\, d\tau_1 = 0 \quad (j = k_1 + 1, \ldots, k), \tag{7}$$

which are, respectively, the conditions for single-valued displacements in tracing the contours of each cut and the conditions of vanishing of the principal vector of the forces acting on each rigid inclusion.

The desired angles of rotation of the rigid inclusions in the loaded plate are determined from the condition that the principal moment of the forces acting on each rigid inclusion must vanish. Using the relations [8]

$$M\Big|_A^B = 2\mathrm{Re}\Big\{\sum_{\nu=1}^{2} F_\nu(z_\nu) - z_\nu \varphi_\nu(z_\nu)\Big\}\Big|_A^B, \quad \Phi_\nu = \frac{d\varphi_\nu}{dz_\nu}, \quad \varphi_\nu = \frac{dF_\nu}{dz_\nu},$$

we write this condition in the form

$$2\mathrm{Re}\Big\{\int_{L_j} (\tau_1 - \tau_2 A_0 - \bar{\tau}_2 \bar{B}_0)\mu_1(\tau)\, d\tau_1\Big\} = 0 \qquad (j = k_1 + 1, \ldots, k). \tag{8}$$

Thus, system (5)–(8) serves to determine the densities $\omega_1(t)$, $\omega_2(t)$, $\mu_1(t)$, and $\mu_2(t)$.

Numerical Solution. Using the parametrized equations of the curves $L_j = \{t = \tau^j(\xi),\ \xi \in [-1, 1]\}$ and introducing the notation

TABLE 1

α	$K_1(a)/(p\sqrt{\pi R})$		$K_2(a)/(p\sqrt{\pi R})$	
	Present work	Data of [1]	Present work	Data of [1]
0	−0.157 10	−0.157 10	0.035 73	0.035 73
$\pi/12$	−0.122 00	−0.122 00	0.093 23	0.093 23
$\pi/6$	−0.074 20	−0.074 24	0.105 70	0.105 70
$\pi/4$	−0.026 64	−0.026 66	0.069 76	0.069 77
$\pi/3$	0.008 00	0.008 00	−0.004 91	−0.004 91
$5\pi/12$	0.020 48	0.020 46	−0.098 32	−0.098 34
$\pi/2$	0.007 38	0.007 37	−0.185 50	−0.185 50

$$\omega_1(\tau^j(\xi)) = \chi_j(\xi) = \chi_j^0(\xi)/\sqrt{1-\xi^2} \qquad (j = 1, \ldots, k_1),$$

$$\mu_1(\tau^j(\xi)) = \chi_j(\xi) = \chi_j^0(\xi)/\sqrt{1-\xi^2} \qquad (j = k_1+1, \ldots, k),$$

we reduce system (5)–(8) to the canonical system of integral equations

$$\sum_{p=1}^{k} \int_{-1}^{1} \left\{ K_1^{jp}(\xi,\eta)\chi_p(\eta) + K_2^{jp}(\xi,\eta)\overline{\chi_p(\eta)} \right\} d\eta = f_j(\xi) \quad (j = 1, \ldots, k),$$

$$\int_{-1}^{1} \chi_j(\eta)(\tau_1^j(\eta))'\, d\eta = 0 \qquad (j = 1, \ldots, k),$$

$$\mathrm{Re}\left\{ \int_{-1}^{1} K^J(\eta)\chi_j(\eta)\, d\eta \right\} = 0 \qquad (j = k_1+1, \ldots, k),$$

where the function $K_1^{jj}(\xi,\eta)$ has Cauchy-type singularities.

The system is solved by the scheme of [3] with the use of quadratures. Then the potentials and stresses can be determined with specified accuracy at any point of the plate [7] and the stress-intensity factors K_1 and K_2 can be calculated at the tips of cracks and inclusions [3]:

$$(\sigma_x, \tau_{xy}, \sigma_y) = 2\mathrm{Re}\left\{ \sum_{\nu=1}^{2} (\mu_\nu^2, -\mu_\nu, 1)\Phi_\nu(z_\nu) \right\}, \quad K_1 = \lim_{\substack{r\to 0\\ \theta=0}} \sigma_n\sqrt{2\pi r}, \quad K_2 = \lim_{\substack{r\to 0\\ \theta=0}} \tau_n\sqrt{2\pi r}.$$

Here r and θ are the polar coordinates with the pole located at the tip and the polar axis directed along the tangent to the curve, $\sigma_n = 0.5(\sigma_x + \sigma_y) + 0.5(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi$, and $\tau_n = -0.5(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi$ (φ is the angle between the normal to the curve at its tip and the x axis).

Examples of Calculations. Below, we consider calculation results obtained for the stress-intensity factors at the tips of cuts and undeformable inclusions in isotropic and anisotropic (orthotropic) plates subjected to uniaxial extension. The following materials are considered: isotropic material for $E = 720$ GPa and $\nu = 0.25$ (No. 1), anisotropic composite for $E_1 = 780$ GPa, $E_1/E_2 = 3$, $G = 120$ GPa, and $\nu = 0.25$ (No. 2), graphite-epoxy composite for $E_1 = 276.1$ GPa, $E_1/E_2 = 25$, $G = 5.52$ GPa, and $\nu = 0.25$ (No. 3), and glass-epoxy composite for $E_1 = 53.84$ GPa, $E_1/E_2 = 3$, $G = 8.63$ GPa, and $\nu = 0.25$ (No. 4). Here E, E_1, and E_2 are the elastic moduli, G is the shear modulus, and ν is the Poisson's ratio. In the calculations, the "weak anisotropy," which is characterized by the ratio $E_1/E_2 = 0.9996$, was introduced to model the isotropic material.

Figures 2–4 and Table 1 show the stress-intensity factors K_1 and K_2 at the tips of a rigid inclusion shaped like a semicircular arc as a function of the angle α, which characterizes the position of this inclusion in the plate upon uniaxial extension.

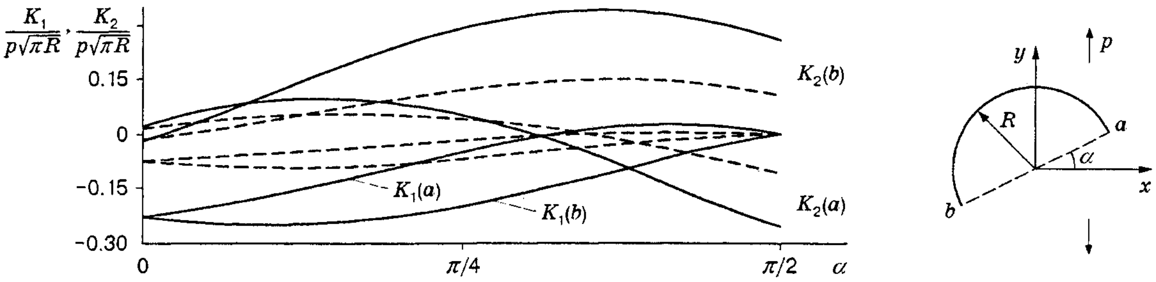

Fig. 2

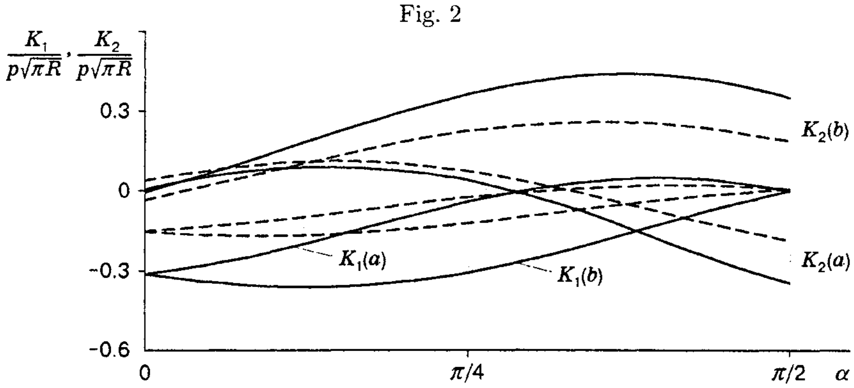

Fig. 3

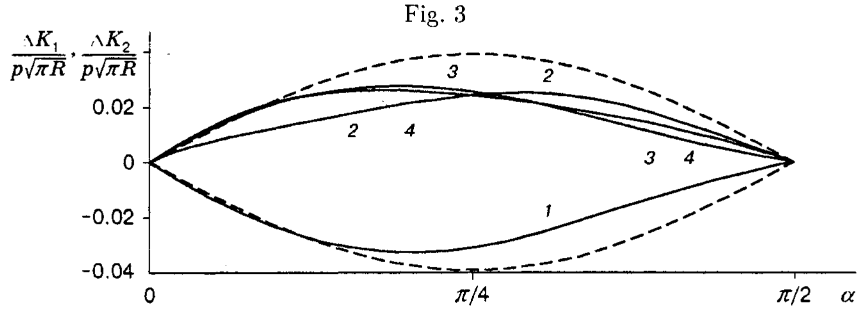

Fig. 4

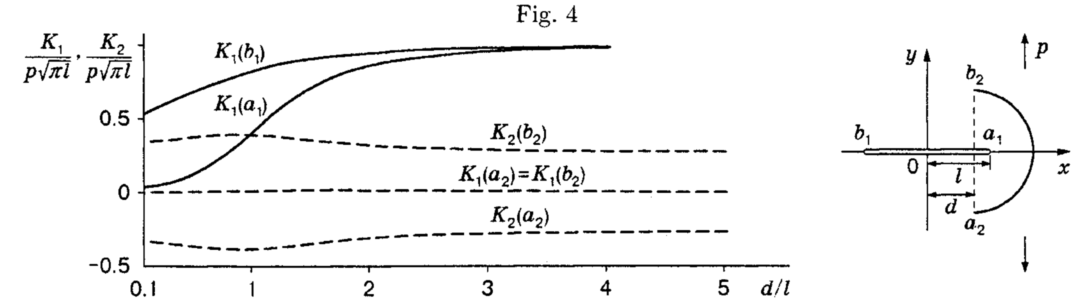

Fig. 5

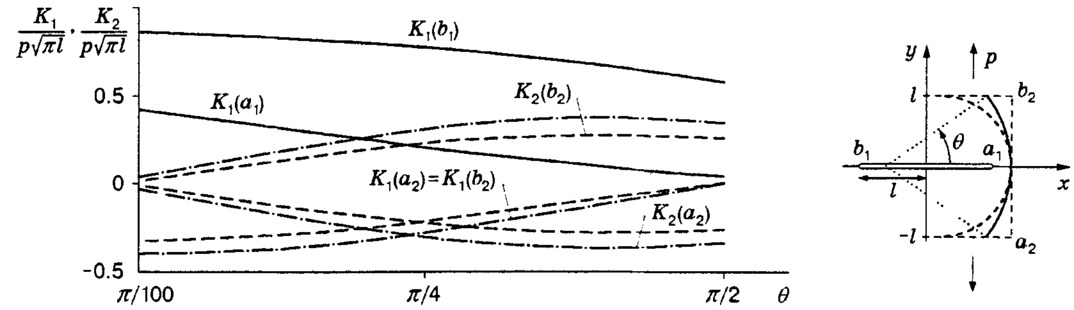

Fig. 6

Figure 2 shows calculation results for anisotropic material No. 2 for the following two cases: 1) the principal direction of the anisotropy coincides with the x direction (solid curves); 2) the principal direction of the anisotropy coincides with the y direction (dashed curves). The results in Fig. 3 refer to material Nos. 1 and 3 (dashed and solid curves, respectively). It follows from Figs. 2 and 3 that, for any α, the stress-intensity factors depend heavily on the anisotropy parameters and the angle between the anisotropy axis and the direction of the tensile load at infinity.

In Fig. 4, the difference between the stress-intensity factors in the problem considered and in the problem in which the rotation of an inclusion is not allowed is plotted against α. The principal direction of the anisotropy coincides with the x direction. The dashed curves refer to the dependences $\Delta K_1(a)/(p\sqrt{\pi R})$ (the lower curve) and the coinciding dependences $\Delta K_2(a)/(p\sqrt{\pi R})$, $\Delta K_1(b)/(p\sqrt{\pi R})$, and $\Delta K_2(b)/(p\sqrt{\pi R})$ (the upper curve) for isotropic material No. 1, solid curves 1–4 refer to $\Delta K_1(a)/(p\sqrt{\pi R})$, $\Delta K_2(a)/(p\sqrt{\pi R})$, $\Delta K_1(b)/(p\sqrt{\pi R})$, and $\Delta K_2(b)/(p\sqrt{\pi R})$, respectively, for anisotropic material No. 3. The symmetry which is typical of isotropic materials is not observed here.

In Table 1, we compare the approximate values of the stress-intensity factors for isotropic material No. 1 (see Fig. 3) with those calculated by the exact formula given in [1] for several values of α. It is clear from Table 1 that the proposed method, in which the concept of "weak anisotropy" is used, can be applied to studying the stress–strain state in isotropic plates.

The stress-intensity factors at the tips of a rectilinear cut of length $2l$ (solid curves) and a rigid inclusion shaped like a semicircumference of radius $R = l$ whose center lies on the straight line passing along the cut (dashed curves) as a function of the distance d are shown in Fig. 5 for anisotropic material No. 4. The plane is subjected to uniaxial extension. One can see that the cut and the inclusion exert a negligible effect on each other for $d/l > 3$, i.e., when the distance between the circumference center and the nearest cut tip is greater than the cut length.

For material No. 4, Fig. 6 shows the stress-intensity factors K_1 and K_2 at the tips of a rectilinear cut of length $2l$ (solid curves) and a rigid inclusion shaped like a circumference arc (dot-and-dashed curves) as a function of the central angle 2θ (as the angle θ changes from 0 to $\pi/2$, the shape of the inclusion changes from a linear segment to a semicircular arc). The dashed curves show the factors K_1 and K_2 versus the angle 2θ for a single rigid inclusion.

It is noteworthy that, for an isotropic material, Berezhnitskii et al. [1] proposed approximate formulas for calculating the stress-intensity factors at the tips of a pair of mutually perpendicular objects, namely, a rectilinear crack and an inclusion, in the form of an expansion in powers of the small parameter $\lambda = 2l/d$. However, a numerical analysis has shown that, for $\lambda \leqslant 1/2$, the objects can already be treated as isolated objects by virtue of the large distance between them (for solitary objects, the values of the factors K_1 and K_2 differ by 2% or more).

In summary, the proposed method allows one to obtain quantitative and qualitative estimates of the stress–strain and limit states of plates with cracks and thin rigid inclusions.

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